

Linear Algebra

Exam

Common part

Fall 2021

Answers

For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 points if your answer is incorrect.

Notation (all standard)

- \mathbb{R} denotes the set of real numbers.
- For a matrix A , $a_{ij} \in \mathbb{R}$ denotes the entry of A in row i and column j .
- For a vector $\mathbf{x} \in \mathbb{R}^n$, x_i denotes the i th coordinate of \mathbf{x} .
- I_m denotes the $m \times m$ identity matrix.
- \mathbb{P}_n is the vector space of polynomials of degree less than or equal to n .
- The scalar or inner product of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$.

First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct response.

Question 1 : The straight line that best approximates (in the sense of least squares) the (x, y) point data $(-2, -4)$, $(0, 0)$, $(1, 4)$, $(2, 1)$, $(4, -1)$ is

☐ $y = \frac{1}{2} - \frac{1}{2}x$

☐ $y = \frac{1}{2} + \frac{1}{2}x$

☒ $y = -\frac{1}{2} + \frac{1}{2}x$

☐ $y = -\frac{1}{2} - \frac{1}{2}x$

Question 2 : The inverse $B = A^{-1}$ of the matrix

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -3 & 4 & 7 \\ 4 & -2 & -7 \end{pmatrix}$$

is such that

☐ $b_{32} = -2$

☒ $b_{32} = 2$

☐ $b_{32} = 1$

☐ $b_{32} = -1$

Question 3 : Let A be a symmetric matrix such that

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

are three eigenvectors of A associated with, respectively, the three eigenvalues -5 , 5 , and 2 . Then

☐ $A = \begin{pmatrix} 5 & -4 & 0 \\ -4 & -5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

☒ $A = \begin{pmatrix} 3 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & -3 \end{pmatrix}$

☐ $A = \begin{pmatrix} 3 & -4 & 0 \\ -4 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

☐ $A = \begin{pmatrix} 5 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & -5 \end{pmatrix}$

Question 4 : For $a \in \mathbb{R}$, the system

$$\begin{cases} x + & ay + 3z = 0 \\ & y - 2z = 3 \\ -x + & 4y + 4z = a \\ & (a + 6)y + 3z = a^2 \end{cases}$$

☒ has at least one solution if and only if $a \in \{-2, 3\}$

☐ has infinitely many solutions if and only if $a = -2$

☐ has no solution if and only if $a \in \{-2, 3\}$

☐ has a unique solution if and only if $a = 3$

Question 5 : Let

$$A = \begin{pmatrix} 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 1 & 5 \\ 1 & 3 & 1 & 2 & 2 \\ 0 & 0 & 5 & 0 & 4 \end{pmatrix}.$$

Then

☐ $\det(A) = 6$ ☒ $\det(A) = -24$ ☐ $\det(A) = 0$ ☐ $\det(A) = 48$

Question 6 : Let R be the reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 1 & 5 & 2 & 5 \\ 3 & 3 & -6 & -2 \end{pmatrix}.$$

Then

☐ $r_{13} = -4$ ☐ $r_{13} = 0$ ☐ $r_{13} = -2$ ☒ $r_{13} = -3$

Question 7 : Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a linear transformation with matrix

$$A = \begin{pmatrix} 0 & 5 & 4 & 4 \\ 2 & 1 & 1 & 5 \\ 4 & 3 & -5 & 3 \end{pmatrix},$$

(with respect to the standard bases for \mathbb{R}^4 and \mathbb{R}^3). Then

- ☐ T is bijective, i.e. both injective and surjective
☐ T is injective but not surjective
☒ T is surjective but not injective
☐ T is neither injective nor surjective

Question 8 : Let \mathcal{E} be the standard basis of \mathbb{R}^4 and $\mathcal{B} = \{1 + t^2, 2 - t^3, t, 1 - t^2\}$ a basis of \mathbb{P}_3 . Let $T : \mathbb{P}_3 \rightarrow \mathbb{R}^4$ be the linear transformation

$$T(a + bt + ct^2 + dt^3) = \begin{pmatrix} a - b \\ a - c \\ 2a + c \\ 2b + d \end{pmatrix}.$$

The matrix A for T with respect to the bases \mathcal{B} of \mathbb{P}_3 and \mathcal{E} of \mathbb{R}^4 , i.e. $[T(p)]_{\mathcal{E}} = A[p]_{\mathcal{B}}$ for all $p \in \mathbb{P}_3$ is such that

☐ $a_{34} = -1$ ☐ $a_{34} = 2$ ☐ $a_{34} = -2$ ☒ $a_{34} = 1$

Question 9 : Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right\}$$

be two bases of \mathbb{R}^3 . Then the change of coordinates matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ from the basis \mathcal{B} to the basis \mathcal{C} , i.e. the matrix such that $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}} [\mathbf{x}]_{\mathcal{B}}$ for all $\mathbf{x} \in \mathbb{R}^3$, is

☐ $\begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 1 \end{pmatrix}$
☒ $\begin{pmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ -1 & -3 & 5 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & -3 \\ -3 & 1 & 5 \end{pmatrix}$

Question 10 : Given

$$A = \begin{pmatrix} 4 & 1 & -8 \\ 1 & 1 & 2 \\ 2 & 1 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix},$$

the solution $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ of $A\mathbf{x} = \mathbf{b}$ has first coordinate

- ☐ $x_1 = -1$ ☐ $x_1 = 5$ ☐ $x_1 = 4$ ☒ $x_1 = -2$

Question 11 : Let

$$A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 20 \\ 0 \end{pmatrix}.$$

Then the least squares solution $\hat{\mathbf{x}} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}$ of $A\mathbf{x} = \mathbf{b}$ satisfies

- ☐ $\hat{x}_1 = -3$ ☐ $\hat{x}_1 = -6$ ☐ $\hat{x}_1 = 6$ ☒ $\hat{x}_1 = 3$

Question 12 : Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The real roots of the characteristic polynomial of A are

- ☐ $\{0\}$ ☐ $\{-1, 0\}$ ☐ $\{0, 1\}$ ☒ $\{-1, 1\}$

Question 13 : The Gram-Schmidt algorithm applied to the columns of the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 1 & -1 & 6 \\ 1 & 1 & 3 \\ 0 & 1 & -6 \end{pmatrix}$$

yields an orthogonal basis of $\text{Col}(A)$ given by the vectors

- ☐ $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ -6 \end{pmatrix}$ ☒ $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ -4 \end{pmatrix}$
- ☐ $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ -4 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \\ -6 \end{pmatrix}$

Question 14 : The orthogonal projection of the vector $\begin{pmatrix} 3 \\ 1 \\ -1 \\ 5 \end{pmatrix}$ on the subspace generated by the first two columns of the matrix A from Question 13 is the vector:

- ☐ $\begin{pmatrix} 21 \\ -15 \\ 3 \\ 9 \end{pmatrix}$ ☐ $\begin{pmatrix} 39 \\ -9 \\ 15 \\ 12 \end{pmatrix}$ ☐ $\begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ ☒ $\begin{pmatrix} 3 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

Question 15 : The matrix A from Question 13 has a QR-decomposition such that

- ☒ $r_{33} = 3\sqrt{2}$ ☐ $r_{33} = 2\sqrt{3}$ ☐ $r_{33} = \sqrt{3}$ ☐ $r_{33} = \sqrt{2}$

Question 16 : Let

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ -5 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 5 \end{pmatrix},$$

and $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\} \subset \mathbb{R}^4$. Then a basis of the orthogonal complement W^\perp is given by the vectors

☐ $\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

☐ $\begin{pmatrix} -6 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

☐ $\begin{pmatrix} -6 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$

☒ $\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$

Second part: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 17 : If $\{v_1, v_2, \dots, v_k\}$ is a linearly independent set of vectors in a vector space V and if $\alpha_1, \alpha_2, \dots, \alpha_k$ are any real numbers, then $\{\alpha_1 v_1, \alpha_2 v_2, \dots, \alpha_k v_k\}$ is also a set of linearly independent vectors in V .

☐ TRUE ☒ FALSE

Question 18 : The polynomials

$$p_1(t) = t^2, \quad p_2(t) = t^2 + t^3, \quad p_3(t) = t - t^3, \quad p_4(t) = 1 + t^3$$

form a basis for \mathbb{P}_3 .

☒ TRUE ☐ FALSE

Question 19 : If A is a 3×3 matrix with eigenvalues 1, 2, and 4, then $\det(A) = 8$.

☒ TRUE ☐ FALSE

Question 20 : If A and B are two $n \times n$ invertible matrices, then AB is invertible and

$$(AB)^{-1} = A^{-1}B^{-1}.$$

☐ TRUE ☒ FALSE

Question 21 : The matrix

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

is diagonalizable.

☒ TRUE ☐ FALSE

Question 22 : The matrix

$$A = \begin{pmatrix} 10 & -11 & 11 \\ 11 & 11 & -11 \\ -11 & 11 & 10 \end{pmatrix}$$

is orthogonally diagonalizable.

☐ TRUE ☒ FALSE

Question 23 : Let A be a 6×6 matrix. If the characteristic polynomial of A is $p(\lambda) = (\lambda - 3)^2 \lambda^4$, then the geometric multiplicity of the eigenvalue $\lambda = 3$ is always 2.

☐ TRUE ☒ FALSE

Question 24 : Let V be a vector space such that $\dim V = n$. If S is a set of linearly independent vectors in V , then S is a basis for V .

☐ TRUE ☒ FALSE

Question 25 : Let A be a 9×5 matrix. If $\dim(\text{Nul}(A)) = 5$, then $\dim(\text{Col}(A)) = 0$.

☒ TRUE ☐ FALSE

Question 26 : Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 5 \\ 3 & 1 & 5 & -1 & -3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Given that the matrix R is the reduced echelon form for the matrix A , then the vectors

$$\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

form a basis for $\text{Nul}(A)$.

☒ TRUE ☐ FALSE

Question 27 : Let A be the matrix from Question 26. Then the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

form a basis for $\text{Col}(A)$.

☐ TRUE ☒ FALSE

Question 28 : Let \mathbf{u} and \mathbf{v} be two vectors in \mathbb{R}^n . If $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$, then \mathbf{u} and \mathbf{v} are orthogonal.

☒ TRUE ☐ FALSE

Question 29 : Let \mathcal{B} be the basis of \mathbb{P}_2 given by $\mathcal{B} = \{1, 1+t, 1+t^2\}$ and let $p \in \mathbb{P}_2$ be $p(t) = 3+4t+5t^2$. Then the first coordinate of p relative to the basis \mathcal{B} is -6 .

☒ TRUE ☐ FALSE

Question 30 : If A is an $n \times n$ matrix, then $\det(A^T) = -\det(A)$.

☐ TRUE ☒ FALSE

Question 31 : The set $\{p \in \mathbb{P}_4 \mid p(t) = at^4 \text{ for some } a \in \mathbb{R}\}$ is a subspace of \mathbb{P}_4 .

☒ TRUE ☐ FALSE

Question 32 : Let

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

Then $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is the closest point in $W = \text{Span}\{\mathbf{u}, \mathbf{v}\} \subset \mathbb{R}^3$ to \mathbf{y} .

☒ TRUE ☐ FALSE

Question 33 : Let V and W be two finite-dimensional vector spaces and $T : V \rightarrow W$ a linear transformation. If $\dim W < \dim V$, then T is not injective.

☒ TRUE ☐ FALSE